5/H–29 (v) (Syllabus–2015)

2 0 1 7

(October)

MATHEMATICS

(Honours)

(Elementary Number Theory and Advanced Algebra)

(GHS–51)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer Elementary Number Theory and Advanced Algebra in two separate books

Answer **five** questions, choosing **one** from each Unit

UNIT—I

(Elementary Number Theory)

1. (a) State whether the following statements are True or False with brief justification (a, b, c, n denote integers) (any *five*) : 2×5=10

(i) $4|(n^2+2)$ for some integer n.

(ii) If $(a,b)=1$ and $c|a$ then $(b,c)=1$.

(Turn Over)

(iii) If $b \mid a^2+1$, then $b \mid a^4+1$.

(iv) The only prime of the form n^3-1 is 7.

(v) If $(n, 7)=1$ then $7 \mid n^6-1$.

(vi) n^2+n+41 is prime for every positive integer n.

(b) Prove that there is an infinite number of primes. 5

2. (a) State and prove Fermat's Little theorem. 1+5=6

(b) What is the last digit in the decimal representation of 3^{100}? 4

(c) Prove that $a^5 \equiv a \pmod{10}$ for every integer a. 5

UNIT—II

3. (a) State and prove Chinese Remainder theorem. 1+4=5

(b) Solve the following system of linear congruences : 5

$$x \equiv 2 \pmod 3$$
$$x \equiv 3 \pmod 5$$
$$x \equiv 2 \pmod 7$$

(Continued)

(c) Prove that $\phi(5n)=5\phi(n)$ if and only if 5 divides n. 5

4. (a) Find the highest power of 2 which divides 533!. 2

(b) Evaluate : 2

$$\sum_{j=1}^{\infty} \mu(j!)$$

(c) Evaluate $\sigma(n)$ and $\tau(n)$ for $n=3000$. 2+2=4

(d) Evaluate $\phi(3125)$. 2

(e) For any real numbers x and y, prove that $[x+y] \le [x]+[y]+1$. 5

UNIT—III

(Advanced Algebra)

5. (a) Prove that the intersection of any two normal subgroups of a group G is a normal subgroup of G. 5

(b) If f is a homomorphism of group G into a group G' with kernel K, then prove that K is a normal subgroup of G. 5

(c) Prove that any finite integral domain is a field. 5

(Turn Over)

6. (a) If R is the additive group of real numbers and R^+ is the multiplicative group of positive real numbers, then show that the mapping $f: R^+ \to R$ such that $f(x) = \log x$; $x \in R^+$ is an isomorphism. 5

(b) If R is a ring such that $a^2 = a$, for all $a \in R$, prove that $a + a = 0$, for all $a \in R$. 2

(c) The set M of 2×2 matrices over the field of real numbers is a ring with respect to matrix addition and multiplication. Does this ring possess zero divisors? Justify your answer. 2

(d) If R is a commutative ring and $a \in R$, then prove that the set $Ra = \{ ra : r \in R \}$ is an ideal of R. 3

(e) Define maximal ideal of a ring R. Is $\{0\}$ in the ring of integers $\mathbb{Z}$ a maximal ideal? Justify your answer. 2+1=3

UNIT—IV

7. (a) Prove that a field has only two ideals 0 and itself. 3

(b) Determine all the ideals in $\mathbb{Z}_6$. 3

(c) Define units. Determine the number of units in the ring of integers. 2+2=4

(d) Consider the ring $\mathbb{Z}$. In this ring $5\mathbb{Z} = \{5k : k \in \mathbb{Z}\}$ is an ideal of $\mathbb{Z}$. How many distinct cosets are there in the quotient ring $\mathbb{Z}/5\mathbb{Z}$? Is this quotient ring a field? Justify your answer. 2+3=5

8. (a) Let $\mathbb{R}$ be an integral domain and a, $b \in \mathbb{R}$. When do we say the following?

(i) a and b are associates in $\mathbb{R}$

(ii) a is an irreducible element in $\mathbb{R}$

(iii) a is a prime element in $\mathbb{R}$ 2×3=6

(b) (i) Show that the polynomial $x^2 + x + 4$ is irreducible over F, the field of integers modulo 11. 3

(ii) Prove that 3 is not a prime element in $\mathbb{Z}[\sqrt{-5}]$. 4

(iii) Find an associate of a non-zero element in $\mathbb{Z}$. 2

(Turn Over)

UNIT—V

9. (a) Let $V(F)$ be a vector space over a field F and let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a set of vectors of V. When is this set of vector said to be linearly independent? Give an example of a finite set of vectors which is linearly independent in the vector space $\mathbb{R}^3$ and $\mathbb{R}$. 2+2=4

(b) Show that the subset {(1, 2, 0), (0, 3, 1) (–1, 0, 1)} of the vector space $V_3(R)$, where R is the field of real numbers, is linearly independent. 5

(c) Prove that each subspace W of a finite dimensional vector space $V(F)$, Fa, field of dimension n is a finite dimensional space with dim $m \leq n$. 6

10. (a) Let T be the linear operator on $\mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$$

What is the matrix of T with respect to the basis $\beta = \{(1,0,0), (0,1,0), (0,0,1)\}$? Show that T is invertible. 3+2=5

(Continued)

(b) Let $T: R^3 \to R^3$ be a linear transformation defined by

$$T(a_1, a_2, a_3) = (a_1, -a_2, 2a_3).$$

Find the null space of T and range T. 6

(c) Let $V = \mathbb{R}^4$ the real vector space and let $S = \{(2,0,0,1), (-1,0,1,0)\}$. Find $L(S)$ i.e., the set of all linear combinations of finite sets of elements of S. 4

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